(N_j) in the following way:

$$N_0 := 0,\ N_j := \sup\{n \in \mathbb{N} : \alpha_n \geq \frac{\alpha_1}{2^j}\},\ j \in \mathbb{N}.$$

Then we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \rho(\alpha_n) &\leq \sum_{j=1}^{\infty} (N_j - N_{j-1}) \rho\left(\frac{\alpha_1}{2^{j-1}}\right) \\
&\leq \sum_{j=1}^{\infty} N_j\, \rho\left(\frac{\alpha_1}{2^{j-1}}\right) \\
&= 2 \int_0^{2\alpha_1} \sum_{j=1}^{\infty} N_j \mathbf{1}_{\left(\frac{\alpha_1}{2^{j-1}}, \frac{\alpha_1}{2^{j-2}}\right]}(t) \cdot \sum_{j=1}^{\infty} \frac{\rho\left(\frac{\alpha_1}{2^{j-1}}\right)}{\frac{\alpha_1}{2^{j-2}}} \mathbf{1}_{\left(\frac{\alpha_1}{2^{j-1}}, \frac{\alpha_1}{2^{j-2}}\right]}(t)\, dt \\
&\leq 2 \int_0^{2\alpha_1} \sum_{j=1}^{\infty} N_j \mathbf{1}_{\left(\frac{\alpha_1}{2^{j-1}}, \frac{\alpha_1}{2^{j-2}}\right]}(t) \cdot \frac{\rho(t)}{t}\, dt \\
&\leq 4 \left\| \sum_{j=1}^{\infty} N_j \mathbf{1}_{\left(\frac{\alpha_1}{2^{j-1}}, \frac{\alpha_1}{2^{j-2}}\right]} \right\|_{L_{q/p}([0,2\alpha_1])} \cdot \|\varkappa\|_{L_{q/(q-p)}([0,2\alpha_1])}.
\end{aligned}$$

Now we notice that

$$\left\| \sum_{j=1}^{\infty} N_j \mathbf{1}_{\left(\frac{\alpha_1}{2^{j-1}}, \sum_{j=1}^{\infty} \varphi(N_j) \frac{\alpha_1}{2^{j-1}} = \frac{\alpha_1}{2^{j-2}}\right]} \right\|_{L_{q/p}([0,2\alpha_1])}$$

$$= \sum_{j=1}^{\infty} (N_j)^{q/p} \frac{\alpha_1}{2^{j+1}} = 4 \sum_{j=1}^{\infty} (N_j)^{q/p} \frac{\alpha_1}{2^{j+1}}$$

$$= 4 \left\| \underbrace{\frac{\alpha_1}{2}, \ldots, \frac{\alpha_1}{2}}_{N_1}, \underbrace{\frac{\alpha_1}{2^2}, \ldots, \frac{\alpha_1}{2^2}}_{N_2 - N_1}, \ldots, \underbrace{\frac{\alpha_1}{2^j}, \ldots, \frac{\alpha_1}{2^j}}_{N_j - N_{j-1}}, \ldots \right\|_{p/q,1} \leq 4 \|(\alpha_n)\|_{p/q,1} \leq 1,$$

so

$$\left\| \sum_{j=1}^{\infty} N_j \mathbf{1}_{\left(\frac{\alpha_1}{2^{j-1}}, \frac{\alpha_1}{2^{j-2}},\right]} \right\|_{L_{q/p}([0,2\alpha_1])} \leq 1,$$

hence,

$$\sum_{n=1}^{\infty} \rho(\alpha_n) \leq 4 \|\varkappa\|_{L_{q/(q-p)}([0,2\alpha_1])} \qquad \square$$